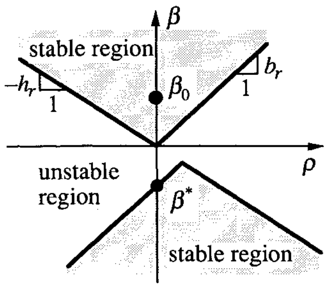

Fig. 2 Stability diagram.

where the various $\Phi(t)$ are the displacement components of the transition matrix for the closed-loop system. The trace indicates that an average is taken over all possible initial states satisfying $|x(0)|=1$ and $|\dot{x}(0)|=0$. Analytical expressions for E were developed for the case $b_1=b_2$; the results are[1] $E=(\beta^2+h^2)/(\beta^2 h)$ for velocity feedback and for pole allocation with $h\leq\beta$, $E=(\beta^6+6h^2\beta^2+h^4)/(4\beta^2h^3)$ for pole allocation with $h>\beta$, and

$$E=\left(\frac{\beta}{h}+\frac{2h}{\beta}\right)\left(2\sqrt{h^2+\beta^2}-\beta\right)^{-1} \tag{7}$$

for optimal control. For all three algorithms, $E\rightarrow\infty$ as $\beta\rightarrow 0$, as expected.

Figure 1b shows how the response index varies with the feedback gain h. For $h<\beta$, the three algorithms are almost equally effective in reducing E. However, for larger gains, velocity feedback and pole allocation actually yield larger E as h increases. Only optimal control yields a response index that decreases monotonically with h. The asymptotic behavior of E for increasing h is[1] $E\sim h/\beta^2$ for velocity feedback, $E\sim h/4\beta^2$ for pole allocation, and $E\sim 1/\beta$ for optimal control. These results were illustrated by a numerical investigation of an antenna mast.[1]

Stability Conditions

If the open-loop parameters of the structure are not known with certainty, then the matrix Λ in Eq. (3) will have unknown frequency spacing β and off-diagonal elements $\rho\omega_a^2$. (Small uncertainties in ω_a are relatively unimportant.) Given a set of control parameters, it has been rigorously shown that the instability condition is[1]

$$\min\{b_r\rho, \beta^*+h_r\rho\}<\beta<\max\{b_r\rho, h_r\rho\} \tag{8}$$

when $\beta^*<0$ and

$$\min\{b_r\rho, h_r\rho\}<\beta<\max\{b_r\rho, \beta^*+h_r\rho\} \tag{9}$$

when $\beta^*\geq 0$. Here, $b_r=(b_1/b_2-b_2/b_1)/4$, $h_r=(h_1/h_2-h_2/h_1)/4$, and $\beta^*=(b_1h_1+b_2h_2)(g_1/h_1-g_2/h_2)/2$ are control-related parameters. (It is assumed that $b_1h_1>0$ and $b_2h_2>0$. This condition makes the diagonal elements of $\boldsymbol{bh}$ positive; it also assures that the average of the closed-loop modal damping ratios is positive.)

The shapes of the unstable and stable regions are shown in the $\rho-\beta$ space in Fig. 2 when $\beta^*<0$. The stable regions are two infinite, cone-shaped areas bounded by rays with slopes b_r and h_r. The point labeled β_0 corresponds to the assumed value for the structural parameters, i.e., the assumed model is $\rho=0$ and $\beta=\beta_0$. If the model is accurate, then the actual parameter value would differ only slightly from the assumed value, and the corresponding point in the $\rho-\beta$ space would be in a small neighborhood of β_0. Conversely, if the model is inaccurate, then the point corresponding to the actual parameter value may be quite far from β_0. The robustness of the structure with respect to modeling errors can be measured by the distance from β_0 to the boundaries of the unstable region. This distance is small if 1) the structure has very closely spaced natural frequencies, i.e., $|\beta_0|$ is small, and 2) either $|h_r|$ or $|b_r|$ is large. For either case, small modeling errors may result in instability. Although β^* governs the size of the unstable region, it does not affect the distance between β_0 and the boundary of the unstable region. These results were illustrated by a numerical investigation of the antenna mast.[1]

Conclusions

The paper has shown the following: 1) The normalized difference of the open-loop natural frequencies β governs the effectiveness and robustness of the control algorithms. 2) Velocity feedback, pole allocation, and optimal control yield responses that are nearly equal for small gains but are significantly different for velocity gains on the order of β or greater. Optimal control consistently yields the lowest responses. 3) The lower bound for the response of optimally controlled structures is on the order of $1/\beta$. Thus, structures with small β (i.e., closely spaced natural frequencies) cannot be effectively controlled by a single control input. 4) It is possible, through pole allocation, to increase the modal damping of the closed-loop system by increasing the feedback gains. However, when the velocity gains are larger than β, this algorithm is ineffective in controlling the response. 5) The stable region is bounded by two cones, with geometry determined by the control parameters. A system always becomes more robust as β increases.

Acknowledgments

This research was supported by the National Science Foundation under Grant BCS-8858549 and by the Ohsaki Research Institute of Shimizu Corporation.

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New Proof of the Jacobi Necessary Condition

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Introduction

LOOSELY speaking, a point along a reference extremal is called a conjugate point, if its state, time coordinate can be reached along a neighboring extremal with equal cost. As a typical example, consider the problem of finding the minimum

Received Oct. 23, 1991; revision received Nov. 27, 1992; accepted for publication Jan. 29, 1993.

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traveling distance between two points located on the surface of a sphere. The first-order necessary conditions of optimal control require this path to be a great circle segment. Traveling along a great circle segment, one always encounters a conjugate point at arc length 180 deg.

The Jacobi necessary condition states that an extremal cannot be optimal if it violates the no-conjugate-point condition. Furthermore, the Jacobi sufficient condition states that, under certain conditions, an extremal furnishes at least a weak local minimum if no conjugate points are present (see Refs. 1-3). Unfortunately, because of its local character and because of its restriction to weak local minima the Jacobi sufficient condition is mostly of theoretical importance. In contrast, the benefits of the Jacobi necessary condition for practical applications are clear (see Refs. 2 and 3). A solution candidate, i.e., an extremal satisfying all of the first-order necessary conditions, can immediately be dismissed as nonoptimal if the Jacobi necessary condition is violated.

Presently all Jacobi testing procedures require the extremal under investigation to be smooth. This condition poses a serious restriction on the results obtained by Jacobi tests. Typically, conjugate points occur for "long" extremals. Hence, by applying the Jacobi necessary condition only to smooth subarcs of a given extremal may result in an essential loss of information.

In this Note, a new proof is given for Jacobi's necessary condition. It is shown that the existence of a conjugate point in the interior of an extremal implies the existence of control perturbations that lead to a reduction in cost.

The analysis in this Note is restricted completely to linear-quadratic optimal control problems (LQP). By virtue of the accessory minimum problem this poses no loss of generality. Important ideas used in this Note are adopted from Breakwell and Ho.[1]

Class of Linear-Quadratic Optimal Control Problems

Before stating the problem treated in this Note we will define the transition matrix and, without proof, give a useful lemma tailored for our purposes (see also, Ref. 4).

Lemma 1: Let $A(t)$ be a continuous matrix function of time with at most finitely many points, say, $t_1, \ldots, t_n$ of discontinuity. Assume that at each point of discontinuity t_i, the left-hand/right-hand limit of $A(t)$ exists and is finite. Then the transition matrix $\Phi(t, t_0)$ associated with $A(t)$ is determined uniquely and is nonsingular for all times, i.e.,

$$\det \Phi(t, t_0) \neq 0$$

where

$$\frac{\mathrm{d}\Phi(t,t_0)}{\mathrm{d}t} = A(t)\Phi(t,t_0)$$

$$\Phi(t_0,t_0) = I$$

In the remainder of this Note we will investigate the following optimal control problem.

Definition 1: Let the LQP be defined by

$$\min \frac{1}{2} x(t_f)^T S x(t_f) + \int_{t_0}^{t_f} \frac{1}{2} x(t)^T Q(t) x(t) + \frac{1}{2} u(t)^T R(t) u(t) \, \mathrm{d}t$$

$$\begin{gathered} \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ x(t_0) = 0 \qquad Tx(t_f) = 0 \end{gathered} \tag{1}$$

with t_0, t_f fixed. Here $S \in \boldsymbol{R}^{n,n}$, $T \in \boldsymbol{R}^{s,n}$, $s \leq n$, are fixed matrices; $A(t) \in \boldsymbol{R}^{n,n}$, $B(t) \in \boldsymbol{R}^{n,m}$, $Q(t) \in \boldsymbol{R}^{n,n}$, and $R(t) \in \boldsymbol{R}^{m,m}$ are time-varying, continuous matrix functions of time with at most finitely many points of discontinuity, all of the type described in Lemma 1. Also

$$\begin{gathered} S^T = S \\ Q(t)^T = Q(t) \quad \forall t \\ R(t)^T = R(t) \quad \forall t \\ \|R(t)\| > r_{\min} \quad \forall t, \qquad \text{for some } 0 < r_{\min} \in \boldsymbol{R} \end{gathered} \tag{2}$$

For later reference we now state the first-order necessary conditions associated with this problem.

Lemma 2: Necessary conditions for a solution of the LQP stated in Definition 1 are that there is an absolutely continuous function of time $\lambda(\cdot)$ and a constant vector $\nu \in \boldsymbol{R}^s$ such that

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix}$$

$$x(t_0) = 0$$

$$Tx(t_f) = 0$$

$$\lambda(t_f) = Sx(t_f) + T^T\nu$$

$$\lambda^+ - \lambda^- = 0 \quad \text{at any point of discontinuity of } A, B, Q, \text{ and } R$$

$$u(t) = -R(t)^{-1}B(t)^T\lambda(t)$$

Proof: See Refs. 2, 5, and 6.

No-Conjugate-Point Condition for Linear Quadratic Optimal Control Problem

Definition 2 (Conjugate Point): Time $t_c \in (t_0, t_f)$ is called a conjugate point for the LQP stated in Definition 1 if there is a nontrivial solution to the boundary value problem (BVP) implied by the stationarity conditions for LQP (this BVP is given in Lemma 2) such that (see Fig. 1)

$$x(t) \equiv 0 \quad \text{on} \quad [t_0, t_c]$$

$$x(t) \neq 0 \quad \text{on} \quad (t_c, t_c + \epsilon], \qquad \text{some } \epsilon > 0$$

Assume the LQP stated in Definition 1 has a conjugate point, say, at time $t_c \in (t_0, t_f)$. Then there are at least two distinct extremals leading from the conjugate point t_c to the prescribed terminal manifold (paths 1 and 2 in Fig. 1). In the next lemma it is shown that the costs associated with these extremal arcs are the same.

Lemma 3: Assume the LQP stated in Definition 1 has a conjugate point, say, at time $t_c \in (t_0, t_f)$. Let J_i be the cost for going from t_c to t_f along path i, $i \in \{1, 2\}$, that is,

$$J_i = \tfrac{1}{2} x_f^T S x_f + \int_{\text{path } i} (\tfrac{1}{2} x^T Q x + \tfrac{1}{2} u^T R u) \, \mathrm{d}t$$

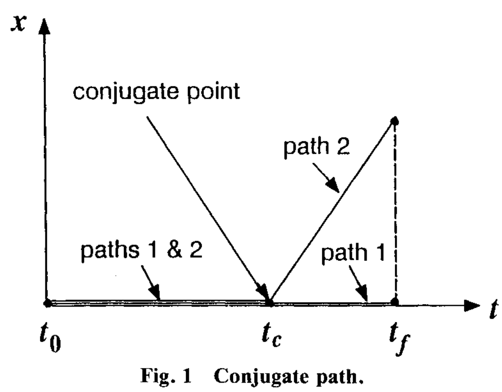

Fig. 1 Conjugate path.

Then $J_1 = J_2 = 0$.

Proof: Trivially, $J_1 = 0$, as $x(t) \equiv 0$ and $u(t) \equiv 0$ along path 1. To compute the cost along path 2 let us denote by $t_1, \ldots, t_k$, $k \geq 0$, all points of discontinuity of the matrix functions of time $A(t)$, $B(t)$, $Q(t)$, and $R(t)$ on the interval $(t_c, t_f]$. Using integration by parts (see Ref. 7) we find

$$
\begin{aligned}
0 &= \int_{t_c}^{t_f} \lambda^T(\dot{x} - Ax + BR^{-1}B^T\lambda)\,\mathrm{d}t \\
&= \int_{t_c}^{t_f} \frac{\mathrm{d}}{\mathrm{d}t}(\lambda^T x) - \dot{\lambda}^T x - \lambda^T(Ax - BR^{-1}B^T\lambda)\,\mathrm{d}t \\
&= \lambda^T x\Big|_{t_c}^{t_1^-} + \lambda^T x\Big|_{t_1^+}^{t_2^-} + \cdots + \lambda^T x\Big|_{t_k^+}^{t_f} - \int_{t_c}^{t_f} \dot{\lambda}^T x \\
&\quad + \lambda^T(Ax - BR^{-1}B^T\lambda)\,\mathrm{d}t \\
&= \lambda^T x\Big|_{t_c}^{t_f} - \int_{t_c}^{t_f} (-Qx - A^T\lambda)^T x + \lambda^T(Ax - BR^{-1}B^T\lambda)\,\mathrm{d}t \\
&= (Sx_f)^T x_f + \int_{t_c}^{t_f} x^T Q x + \lambda^T BR^{-1}B^T\lambda\,\mathrm{d}t \\
&= x_f S x_f + \int_{t_c}^{t_f} x^T Q x + u^T R u\,\mathrm{d}t \\
&= 2J_2 \qquad \text{Q.E.D.}
\end{aligned}
$$

We are now ready to prove the main result of this Note.

Theorem 1: Assume the LQP stated in Definition 1 has a conjugate point, say, at time $t_c \in (t_0, t_f)$. Furthermore, assume that on every subinterval $[t', t''] \subseteq [t_0, t_f]$, $t'' > t'$, the controllability matrix

$$K(t', t'') := \int_{t'}^{t''} \Phi(t, t')^{-1} B(t) B(t)^T \Phi(t, t')^{-T}\,\mathrm{d}t \tag{3}$$

is nonsingular, i.e., the dynamical system $\dot{x} = Ax + Bu$ is controllable on each subinterval $[t', t''] \subseteq [t_0, t_f]$. Then there is a control $\tilde{u}(t)$ which yields negative cost for the LQP and hence the trivial solution $x^0(t) \equiv 0$, $u^0(t) \equiv 0$ (which yields cost $J = 0$) is not optimal. In Eq. (3), $\Phi(t, t')$ denotes the transition matrix associated with matrix $A(t)$ and initial time t' as defined in Lemma 1.

Proof: Consider Fig. 2. Let path 1: $A \to B \to C$, "trivial path," state x^0, control u^0; and path 2: $A \to B \to D$, "conjugate path," state $\bar{x}$, control $\bar{u}$. In Lemma 3 we have seen that the costs associated with path 1 and path 2 are the same. Hence, to show that path 1 is not optimal it suffices to show that path 2 is not optimal.

By assumption, the matrix functions of time $A(t)$, $B(t)$, $Q(t)$, and $R(t)$ have at most finitely many discontinuities. Hence, it is possible to find real numbers $\Delta > 0$, $\delta > 0$, such that $A(t)$, $B(t)$, $Q(t)$, and $R(t)$ are continuous on $[t_c - \Delta, t_c) \cup (t_c, t_c + \delta]$. Here t_c may still be a point of discontinuity.

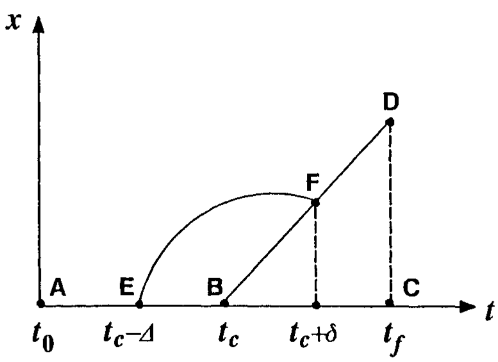

Fig. 2 Construction of an extremal with negative cost.

Additionally, choose $\delta > 0$ small enough such that $x(t) \neq 0$ on $(t_c, t_c + \delta]$ (this is always possible by virtue of the definition of a conjugate point given in Definition 2) and define $x(t_c + \delta) =: \hat{x}_F$.

Now, keeping δ, Δ fixed, consider the optimal control problem

$$\min \int_{t_c - \Delta}^{t_c + \delta} \frac{1}{2} x(t)^T Q(t) x(t) + \frac{1}{2} u(t)^T R(t) u(t)\,\mathrm{d}t \tag{4}$$

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{5}$$

$$x(t_c - \Delta) = 0 \tag{6}$$

$$x(t_c + \delta) = \hat{x}_F \tag{7}$$

Two cases have to be distinguished, namely, 1) problem equations (4–7) do not have a solution and 2) problem equations (4–7) do have a solution.

Case 1. If problem equations (4–7) do not have a solution then especially the conjugate path $\bar{u}(t)$, $\bar{x}(t)$ does not furnish a minimum to the cost criterion

$$J[u] := \int \frac{1}{2} x^T Q x + \frac{1}{2} u^T R u\,\mathrm{d}t$$

along the time interval $[t_c - \Delta, t_c + \delta]$. By virtue of the principle of optimality this implies that the conjugate path, path 2, is not optimal on the interval $[t_0, t_f]$.

Case 2. If problem equations (4–7) do have a solution, say $u^*(t)$, $x^*(t)$, then this solution satisfies the first-order necessary conditions. To show that the conjugate path $\bar{x}(t)$, $\bar{u}(t)$ cannot be optimal on $[t_c - \Delta, t_c + \delta]$ it suffices to show that $x^*(t) \equiv 0$ on any subinterval $[t', t''] \subset [t_c - \Delta, t_c + \delta]$, $t'' > t'$ is not possible [note that along the conjugate path the state is identically zero on the interval $(t_c - \Delta, t_c)$]. The optimality conditions associated with problem Eqs. (4–7) are given by

$$\begin{bmatrix} \dot{x}^* \\ \dot{\lambda}^* \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x^* \\ \lambda^* \end{bmatrix} \tag{8}$$

$$x^*(t_c - \Delta) = 0 \tag{9}$$

$$x^*(t_c + \delta) = \hat{x}_f \tag{10}$$

$$u^*(t) = -R(t)^{-1} B(t)^T \lambda^*(t) \tag{11}$$

The assumed existence of a solution to problem equations (4–7) implies the existence of a solution to the boundary value problem (8–10). Now assume that there is a nonzero time interval $[t', t''] \subset [t_c - \Delta, t_c + \delta]$ with $x^*(t) \equiv 0$ on $[t', t'']$. Then we have on $[t', t'']$

$$
\begin{aligned}
0 &\equiv \dot{x}^* \\
&\equiv A\underbrace{x^*}_{\equiv 0} - BR^{-1}B^T\lambda^* \\
&\equiv -BR^{-1}B^T\lambda^*
\end{aligned}
$$

As $\|R\| > r_{\min} > 0$ on $[t', t'']$ [see Eq. (2)] this implies

$$0 \equiv B^T\lambda^* \qquad \text{on} \qquad [t', t''] \tag{12}$$

Now let $\Phi(t, t')$ denote the transition matrix associated with $A(t)$ and initial time t'. Then $\Phi(t, t')^{-T}$ is the transition matrix associated with $-A(t)^T$ and initial time t'. Using $x^* \equiv 0$, the solution of the costate equation in Eq. (8) is then obtained as $\lambda^*(t) = \Phi(t, t')^{-T}\lambda^*(t')$. Now condition (12) can be rewritten as $0 \equiv B(t)^T \Phi(t, t')^{-T} \lambda^*(t')$ on $[t', t'']$. But this immediately implies that

$$K(t', t'')\lambda^*(t') = 0 \tag{13}$$

where $K(t',t'')$ is the controllability matrix associated with the time interval $[t',t'']$ as defined in Eq. (3). By assumption $K(t',t'')$ is nonsingular. Hence Eq. (13) implies $\lambda^*(t')=0$. But the initial conditions $x^*(t')=0$, $\lambda^*(t')=0$ for the state/adjoint system [Eq. (8)] in conjunction with the assumed continuity of all of the participating matrix functions of time A, B, Q, and R on $[t_c-\Delta, t_c)\cup(t_c, t_c+\delta]$ immediately imply that $x^*(t)\equiv 0$, $\lambda^*(t)\equiv 0$ on $[t_c-\Delta, t_c+\delta]$, even if A, B, Q, and R are not continuous across t_c. But this contradicts $x^*(t_c+\delta) = \hat{x}_F \neq 0$. Hence path 2 is not optimal. Q.E.D.

Summary

A new proof is presented for Jacobi's no-conjugate-point necessary condition. This is achieved by deriving a no-conjugate-point condition for a certain class of LQP. Through the concept of the accessory minimum problem, the result can be generalized to nonlinear optimal control problems. In contrast to earlier results, the new proof also applies if the coefficient functions of time associated with the accessory minimum problem have any finite number of discontinuities.

Acknowledgments

This work was supported in part by NASA Grant NAG-1-946 and NASA Contract NAS1-18935, both with Dan Moereder as technical monitor, and in part by the Air Force Office of Scientific Research (AFOSR) under Grant F49620-93-0280.

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Riccati Solution for the Minimum Model Error Algorithm

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Introduction

CONVENTIONAL filter/smoother algorithms, such as the Kalman filter,[1] require detailed knowledge of both the model error and the measurement error. In nearly all circumstances, the measurement error characteristics are known a priori; however, the errors in the actual system are generally unknown. Also, the Kalman filter assumes that the model error is modeled by a Gaussian noise process. In many cases, such as nonlinearities in the actual system response, this assumption can lead to severely degraded state estimates. The minimum model error[2] (MME) algorithm provides a method of determining optimal state estimation in the presence of significant error in the assumed (nominal) model. The advantages of this method are: 1) the model error and process noise are assumed unknown and are estimated as part of the solution, 2) the model error may take any form (even nonlinear), and 3) the algorithm is robust in the presence of high noise measurements. In several previous studies, this algorithm has been successfully applied to numerous applications, including nonlinear estimation[3] and robust realization/identification of mode shapes in damped structures.[4,5]

The determination of the optimal state estimates using the MME algorithm is derived from a minimization of a cost functional subject to differential equation constraints. In this Note, a closed-form solution to the two-point boundary value problem (TPBVP), associated with the MME algorithm, is developed for linear time-variant state-space models. The closed-form solution is derived using an inhomogeneous Riccati transformation. The resulting forms include one nonlinear Riccati equation and one linear differential equation, each with discrete updates at every output measurement interval.

Minimum Model Error Algorithm

In this section, the MME algorithm is briefly reviewed for the case of linear time-variant state-space models. A more detailed derivation of the algorithm may be found in Ref. 2. The MME algorithm assumes that the state estimates are given by a nominal (prespecified) model and an unmodeled disturbance vector, shown as

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) + d(t) \\ y(t) &= C(t)x(t) + D(t)u(t) \end{aligned} \tag{1}$$

where $A(t)$, $B(t)$, $C(t)$, and $D(t)$ are time-variant nominal state matrices, $u(t)$ is a known forcing input, $d(t)$ is an $(n \times 1)$ unmodeled (to-be-determined) error vector, $x(t)$ is the $(n \times 1)$ state estimate vector, and $y(t)$ is the $(m \times 1)$ estimated output. For the remainder of this Note, the state-space (model) matrices are assumed time variant but are shown without the time argument (t).

State-observable discrete time-domain measurements are assumed for Eq. (1) in the following form:

$$\tilde{y}(t_k) = g_k[x(t_k), t_k] + \nu_k \tag{2}$$

where $\tilde{y}(t_k)$ is an $(m \times 1)$ measurement vector at time t_k, g_k is an accurate model of the measurement process, ν_k represents measurement noise, and m is the total number of measurement output sets. The measurement noise process is assumed to be a zero-mean, Gaussian distributed process of known covariance R.

In the MME, the optimal state estimates are determined on the basis that the measurement-minus-estimate error covariance matrix must match the measurement-minus-truth error covariance matrix. This condition is referred to as the covariance constraint, approximated by

$$\left\{[\tilde{y}(t_k) - y(t_k)][\tilde{y}(t_k) - y(t_k)]^T\right\} \approx R \tag{3}$$

Therefore, the estimated measurements are required to fit the actual measurements with approximately the same error covariance as the actual measurements fit the truth.

A cost functional, consisting of the weighted sum square of the measurement-minus-estimate residuals plus the weighted sum square of the model correction term, is next minimized

$$J = \sum_{k=1}^{m} \left\{[\tilde{y}(t_k) - y(t_k)]^T R^{-1} [\tilde{y}(t_k) - y(t_k)]\right\} + \int_{t_0}^{t_f} d(\tau)^T W d(\tau)\, \mathrm{d}\tau \tag{4}$$

Received June 12, 1992; revision received Nov. 7, 1992; accepted for publication Jan. 29, 1993.

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